

# SOME INVERSE PROBLEMS IN PERIODIC HOMOGENIZATION OF HAMILTON-JACOBI EQUATIONS

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**ABSTRACT.** We look at the effective Hamiltonian  $\bar{H}$  associated with the Hamiltonian  $H(p, x) = H(p) + V(x)$  in the periodic homogenization theory. Our central goal is to understand the relation between  $V$  and  $\bar{H}$ . We formulate some inverse problems concerning this relation. Such type of inverse problems are in general very challenging. In the paper, we discuss several special cases in both convex and nonconvex settings.

## 1. INTRODUCTION

**1.1. Setting of the inverse problem.** For each  $\varepsilon > 0$ , let  $u^\varepsilon \in C(\mathbb{R}^n \times [0, \infty))$  be the viscosity solution to the following Hamilton-Jacobi equation

$$(1.1) \quad \begin{cases} u_t + H(Du^\varepsilon, \frac{x}{\varepsilon}) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u^\varepsilon(x, 0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

The Hamiltonian  $H = H(p, x) \in C(\mathbb{R}^n \times \mathbb{R}^n)$  satisfies

- (H1)  $x \mapsto H(p, x)$  is  $\mathbb{Z}^n$ -periodic,
- (H2)  $p \mapsto H(p, x)$  is coercive uniformly in  $x$ , i.e.,

$$\lim_{|p| \rightarrow +\infty} H(p, x) = +\infty \quad \text{uniformly for } x \in \mathbb{R}^n,$$

and the initial data  $g \in \text{BUC}(\mathbb{R}^n)$ , the set of bounded, uniformly continuous functions on  $\mathbb{R}^n$ .

It was proved by Lions, Papanicolaou and Varadhan [15] that  $u^\varepsilon$ , as  $\varepsilon \rightarrow 0$ , converges locally uniformly to  $u$ , the solution of the effective equation,

$$(1.2) \quad \begin{cases} u_t + \bar{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

The effective Hamiltonian  $\bar{H} : \mathbb{R}^n \rightarrow \mathbb{R}$  is determined by the cell problems as follows. For any  $p \in \mathbb{R}^n$ , we consider the following cell problem

$$(1.3) \quad H(p + Dv, x) = c \quad \text{in } \mathbb{T}^n,$$

where  $\mathbb{T}^n$  is the  $n$ -dimensional torus  $\mathbb{R}^n/\mathbb{Z}^n$ . We here seek for a pair of unknowns  $(v, c) \in C(\mathbb{T}^n) \times \mathbb{R}$  in the viscosity sense. It was established in [15] that there exists

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a unique constant  $c \in \mathbb{R}$  such that (1.3) has a solution  $v \in C(\mathbb{T}^n)$ . We then denote by  $\overline{H}(p) = c$ .

In this paper, we always consider the Hamiltonian  $H$  of the form  $H(p, x) = H(p) + V(x)$ . Our main goal is to study the relation between the potential energy  $V$  and the effective Hamiltonian  $\overline{H}$ . In the case where  $H$  is uniformly convex, Concorde [5, 6] provided some first general results on the properties of  $\overline{H}$ , which is convex in this case. In particular, she achieved some representation formulas of  $\overline{H}$  by using optimal control theory and showed that  $\overline{H}$  has a flat part under some appropriate conditions on  $V$ . The connection between properties of  $\overline{H}$  and weak KAM theory can be found in E [9], Evans and Gomes [10], Fathi [12] and the references therein. We refer the readers to Evans [11, Section 5] for a list of interesting viewpoints and open questions. To date, deep properties of  $\overline{H}$  are still not yet well understood.

In the case where  $H$  is not convex, there have been not so many results on qualitative and quantitative properties of  $\overline{H}$ . Very recently, Armstrong, Tran and Yu [1, 2] studied nonconvex stochastic homogenization and derived qualitative properties of  $\overline{H}$  in the general one dimensional case, and in some special cases in higher dimensional spaces. The general case in higher dimensional spaces is still out of reach.

We present here a different question concerning the relation between  $V$  and  $\overline{H}$ . In its simplest way, the question can be thought of as: how much can we recover the potential energy  $V$  provided that we know  $H$  and  $\overline{H}$ ? More precisely, we are interested in the following inverse type problem:

**Question 1.1.** *Let  $H \in C(\mathbb{R}^n)$  be a given coercive function, and  $V_1, V_2 \in C(\mathbb{R}^n)$  be two given potential energy functions which are  $\mathbb{Z}^n$ -periodic. Set  $H_1(p, x) = H(p) + V_1(x)$  and  $H_2(p, x) = H(p) + V_2(x)$  for  $(p, x) \in \mathbb{R}^n \times \mathbb{R}^n$ . Suppose that  $\overline{H}_1$  and  $\overline{H}_2$  are two effective Hamiltonians corresponding to the two Hamiltonians  $H_1$  and  $H_2$  respectively. If*

$$\overline{H}_1 \equiv \overline{H}_2,$$

*then what can we conclude about the relations between  $V_1$  and  $V_2$ ? Especially, can we identify some common “computable” properties shared by  $V_1$  and  $V_2$ ?*

To the best of our knowledge, such kind of questions have never been explicitly stated and studied before. This is closely related to the exciting projects of going beyond the well-posedness of the homogenization and understanding deep properties of the effective Hamiltonian, which are in general very hard. In this paper, we discuss several special cases in high dimensional spaces and provide detailed analysis in one dimensional space for both convex and nonconvex  $H$ . Some first results for the viscous case are also studied.

## 1.2. Main Results.

### 1.2.1. Dimension $n \geq 1$ .

**Theorem 1.1.** *Assume  $V_2 \equiv 0$ . Suppose that there exists  $p_0 \in \mathbb{R}^n$  such that  $H \in C(\mathbb{R}^n)$  is differentiable at  $p_0$  and  $DH(p_0)$  is an irrational vector, i.e.,*

$$DH(p_0) \cdot m \neq 0 \quad \text{for all } m \in \mathbb{Z}^n \setminus \{0\}.$$

Then

$$\overline{H}_1(p_0) = \overline{H}_2(p_0) \quad \text{and} \quad \min_{\mathbb{R}^n} \overline{H}_1 = \min_{\mathbb{R}^n} \overline{H}_2 \quad \Rightarrow \quad V_1 \equiv 0.$$

In particular,

$$(1.4) \quad \overline{H}_1 \equiv \overline{H}_2 \quad \Rightarrow \quad V_1 \equiv 0.$$

Note that we do not assume  $H$  is convex in the above theorem. As  $V_2 \equiv 0$ , it is clear that  $\overline{H}_2 = H$ . The theorem infers that if  $\overline{H}_1(p_0) = H(p_0)$ ,  $\min_{\mathbb{R}^n} \overline{H}_1 = \min_{\mathbb{R}^n} H$  and  $DH(p_0)$  is an irrational vector, then in fact  $V_1 \equiv 0$ . The requirement on  $DH(p_0)$  seems technical on the first hand, but it is, in fact, optimal. If the set

$$G = \{DH(p) : H \text{ is differentiable at } p \text{ for } p \in \mathbb{R}^n\}$$

only contains rational vectors, (1.4) might fail. See Remark 2.1.

If neither  $V_1$  nor  $V_2$  is constant, the situation usually involves complicated dynamics and becomes much harder to analyze. In this paper, we establish some preliminary results. A vector  $Q \in \mathbb{R}^n$  satisfies a Diophantine condition if there exist  $C, \alpha > 0$  such that

$$|Q \cdot k| \geq \frac{C}{|k|^\alpha} \quad \text{for any } k \in \mathbb{Z}^n \setminus \{0\}.$$

**Theorem 1.2.** Assume that  $V_1, V_2 \in C^\infty(\mathbb{T}^n)$ .

- (1) Suppose that  $H \in C^2(\mathbb{R}^n)$ ,  $\sup_{\mathbb{R}^n} \|D^2 H\| < +\infty$  and  $H$  is superlinear. Then for  $i = 1, 2$  and any vector  $Q \in \mathbb{R}^n$  satisfying a Diophantine condition,

$$(1.5) \quad \int_{\mathbb{T}^n} V_i dx = \lim_{\lambda \rightarrow +\infty} (\overline{H}_i(\lambda P_\lambda) - H(\lambda P_\lambda)).$$

Here  $P_\lambda \in \mathbb{R}^n$  is chosen such that  $DH(\lambda P_\lambda) = \lambda Q$ . In particular,

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Rightarrow \quad \int_{\mathbb{T}^n} V_1 dx = \int_{\mathbb{T}^n} V_2 dx.$$

- (2) Suppose that  $H(p) = \frac{1}{2}|p|^2$ . We have that, for  $i = 1, 2$  and any irrational vector  $Q \in \mathbb{R}^n$ ,

$$(1.6) \quad \int_{\mathbb{T}^n} V_i dx = \lim_{\lambda \rightarrow +\infty} \left( \overline{H}_i(\lambda Q) - \frac{1}{2} \lambda^2 |Q|^2 \right)$$

and

$$(1.7) \quad \lim_{\lambda \rightarrow +\infty} \left( \lambda^2 |Q|^2 - \max_{q \in \partial \overline{H}_i(\lambda Q)} q \cdot \lambda Q \right) = 0$$

- (3) Suppose that  $H(p) = \frac{1}{2}|p|^2$ . If there exists  $\tau > 0$  such that

$$(1.8) \quad \sum_{k \in \mathbb{Z}^n} (|\lambda_{k1}|^2 + |\lambda_{k2}|^2) e^{|k|^{n+\tau}} < +\infty,$$

then

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Rightarrow \quad \int_{\mathbb{T}^n} |V_1|^2 dx = \int_{\mathbb{T}^n} |V_2|^2 dx.$$

Here  $\{\lambda_{ki}\}_{k \in \mathbb{Z}^n}$  are Fourier coefficients of  $V_i$ .

**Remark 1.1.** Due to the stability of the effective Hamiltonian, (1.5) and (1.6) still hold when  $V_1, V_2 \in C(\mathbb{T}^n)$ . The equality (1.6) is essentially known in case  $Q$  satisfies a Diophantine condition. The average of the potential function is the constant term in the asymptotic expansion. See [3, 7, 8] for instance.

Moreover, when  $H(p) = \frac{1}{2}|p|^2$ , if  $V_1$  and  $V_2$  are both smooth, through direct computations of the asymptotic expansions,  $\overline{H}_1 \equiv \overline{H}_2$  leads to a series of identical quantities associated with  $V_1$  and  $V_2$ , which involve complicated combinations of Fourier coefficients. It is very difficult to calculate those quantities and our goal is to extract some new computable quantities from those almost uncheckable ones. The above theorem says that the average and the  $L^2$  norm of the potential can be recovered. See (2.28) for an explicit formula to compute the  $L^2$  norm. The fast decay condition (1.8) is a bit restrictive at this moment. It can be slightly relaxed if we transform the problem into the classical moment problem and apply Carleman's condition.

In fact, we conjecture that the distribution of the potential function should be determined by the effective Hamiltonian under reasonable assumptions. When  $n = 1$ , this is proved in Theorem 1.3 for much more general Hamiltonians. High dimensions will be studied in a future work.

1.2.2. *One dimensional case.* When  $n = 1$ , we have a much clearer understanding of this inverse problem. Let us first define some terminologies.

**Definition 1.1.** We say that  $V_1$  and  $V_2$  have the same distribution if

$$\int_0^1 f(V_1(x)) dx = \int_0^1 f(V_2(x)) dx$$

for any  $f \in C(\mathbb{R})$ .

**Definition 1.2.**  $H : \mathbb{R} \rightarrow \mathbb{R}$  is called strongly superlinear if there exists  $a \in \mathbb{R}$  such that the restriction of  $H$  to  $[a, +\infty)$  ( $H|_{[a, +\infty)} : [a, +\infty) \rightarrow \mathbb{R}$ ) is smooth, strictly increasing, and

$$(1.9) \quad \lim_{x \rightarrow +\infty} \frac{\psi^{(k)}(x)}{\psi^{(k-1)}(x)} = 0 \quad \text{for all } k \in \mathbb{N}.$$

Here  $\psi = \psi^{(0)} = (H|_{[a, +\infty)})^{-1} : [H(a), +\infty) \rightarrow [a, +\infty)$  and  $\psi^{(k)}$  is the  $k$ -th derivative of  $\psi$  for  $k \in \mathbb{N}$ .

Note that condition (1.9) is only about the asymptotic behavior at  $+\infty$ . There is a large class of functions satisfying the above condition, e.g.  $H(p) = e^p$ ,  $H(p) = (c + |p|)^\gamma$  for  $p \in [a, +\infty)$  for any  $a \in \mathbb{R}$ ,  $\gamma > 1$  and  $c \geq 0$ . As nothing is required for the behavior of  $H$  in  $(-\infty, a)$  (except coercivity at  $-\infty$ ),  $H$  clearly can be nonconvex.

**Theorem 1.3.** Assume  $n = 1$  and  $V_1, V_2 \in C(\mathbb{T})$ . Then the followings hold:

(1) If  $H$  is quasi-convex, then

$$V_1 \text{ and } V_2 \text{ have the same distribution} \quad \Rightarrow \quad \overline{H}_1 \equiv \overline{H}_2.$$

(2) If  $H$  is strongly superlinear, then

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Rightarrow \quad V_1 \text{ and } V_2 \text{ have the same distribution.}$$

When  $H$  is nonconvex, statement (1) in the above theorem is not true in general. In order to discuss about the general nonconvex situation, we first need some preparations.

Let us look at a basic nonconvex example of the Hamiltonians as following. This is a typical example of a nonconvex Hamiltonian with non-symmetric wells. Choose  $F : [0, \infty) \rightarrow \mathbb{R}$  to be a continuous function satisfying that (see Figure 1)

- (i) there exist  $0 < \theta_3 < \theta_2 < \theta_1$  such that

$$F(0) = 0, \quad F(\theta_2) = \frac{1}{2}, \quad F(\theta_1) = F(\theta_3) = \frac{1}{3},$$

and  $\lim_{r \rightarrow +\infty} F(r) = +\infty$ ,

- (ii)  $F$  is strictly increasing on  $[0, \theta_2]$  and  $[\theta_1, +\infty)$ , and  $F$  is strictly decreasing on  $[\theta_2, \theta_1]$ .

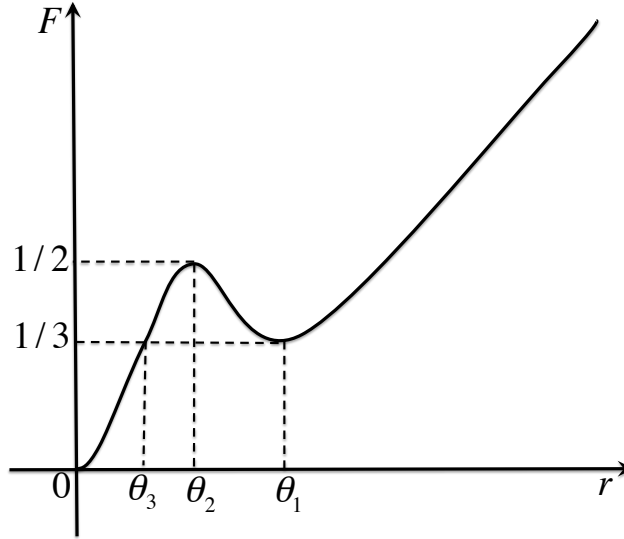


FIGURE 1. Graph of  $F$

The nonconvex Hamiltonian we will use intensively is  $F(|p|)$ .

For  $s \in (0, 1)$ , denote  $V_s : [0, 1] \rightarrow \mathbb{R}$  (see the left graph of Figure 2)

$$(1.10) \quad V_s(x) = \begin{cases} -\frac{x}{s} & \text{for } x \in [0, s], \\ \frac{x-1}{1-s} & \text{for } x \in [s, 1], \end{cases}$$

and extend  $V_s$  to  $\mathbb{R}$  in a periodic way. Let  $H_s(p, x) = H(p) + V_s(x)$  for  $(p, x) \in \mathbb{R}^n \times \mathbb{R}^n$ . Denote by  $\overline{H}_s$  the effective Hamiltonian corresponding to  $H_s$ .

**Definition 1.3.** We say that  $V_1, V_2 \in C(\mathbb{T})$  are macroscopically indistinguishable if

$$\overline{H}_1 \equiv \overline{H}_2$$

for any coercive continuous Hamiltonian  $H : \mathbb{R} \rightarrow \mathbb{R}$ .

Let  $\hat{V} : [0, 1] \rightarrow \mathbb{R}$  be a piecewise linear function oscillating between 0 and -1 (see the right graph of Figure 2) such that

- there exist  $0 = a_1 < c_1 < a_2 < \cdots < a_{m-1} < c_{m-1} < a_m = 1$  for some  $m \geq 2$  and

$$\hat{V}(c_i) = -1 \quad \text{and} \quad \hat{V}(a_i) = 0,$$

- $\hat{V}$  is linear within intervals  $[a_i, c_i]$  and  $[c_i, a_{i+1}]$  for  $i = 1, 2, \dots, m-1$ .

Extend  $\hat{V}$  to  $\mathbb{R}$  in a periodic way.

**Theorem 1.4.** *Let  $s \in (0, 1)$ ,  $V_1 = \hat{V}$ ,  $V_2 = V_s$ .*

- (1)  *$V_1$  and  $V_2$  are macroscopically indistinguishable if*

$$(1.11) \quad \sum_{i=1}^{m-1} \frac{(c_i - a_i)}{s} = \sum_{i=1}^{m-1} \frac{(a_{i+1} - c_i)}{1 - s}.$$

- (2) *For  $H(p) = F(|p|)$ ,*

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Rightarrow \quad (1.11) \text{ holds.}$$

**Remark 1.2.** *From the above theorem, we have that, for  $H(p) = F(|p|)$  and  $s, s' \in (0, 1)$ ,  $\overline{H}_{s'} \equiv \overline{H}_s$  if and only if  $s = s'$ . This demonstrates a subtle difference between convex and non-convex case. If  $H : \mathbb{R} \rightarrow \mathbb{R}$  is convex and even, then the effective Hamiltonian  $\overline{H}$  associated with  $H(p) + V(x)$  for any  $V \in C(\mathbb{T}^n)$  is also even. However, this symmetry breaks down for the non-convex Hamiltonian  $H(p) = F(|p|)$  since*

$$\overline{H}_s(p) = \overline{H}_{1-s}(-p) \neq \overline{H}_s(-p) \quad \text{if } s \neq \frac{1}{2}.$$

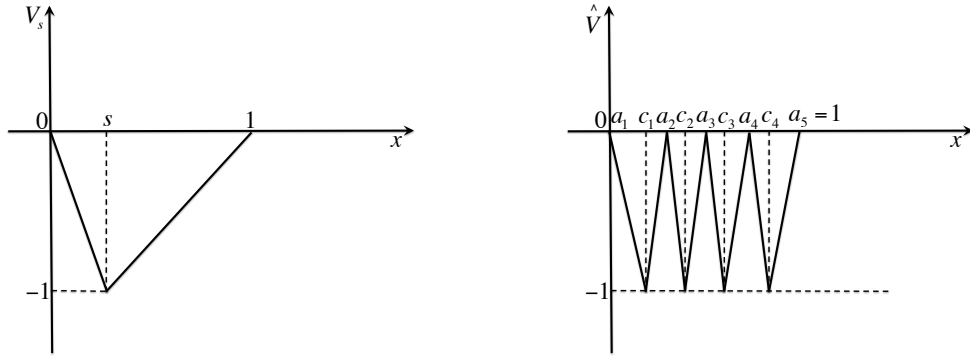


FIGURE 2. Left: Graph of  $V_s$ . Right: Graph of  $\hat{V}$  in case  $m = 5$ .

It is worth mentioning that (1.11) is actually equivalent to the fact that

$$(1.12) \quad \sum_{i=1}^{m-1} (c_i - a_i) = s \quad \text{and} \quad \sum_{i=1}^{m-1} (a_{i+1} - c_i) = 1 - s.$$

The relation (1.12) says that the total length of the intervals where  $\hat{V}$  is decreasing is  $s$  and the total length of the intervals where  $\hat{V}$  is increasing is  $1 - s$ . The assertion of Theorem 1.4 therefore means that,  $\hat{V}$  and  $V_s$  are macroscopically indistinguishable if and only if the total lengths of increasing of  $\hat{V}$  and  $V_s$  are the same and the total lengths of decreasing of  $\hat{V}$  and  $V_s$  are the same. In other words, the above means

that the distribution of the increasing parts of  $\hat{V}$  and  $V_s$  are the same, and so are the decreasing parts.

We note that the requirement that  $\hat{V}$  is piecewise linear is just for simplicity. See Theorem 3.1 for a more general result.

The requirement that  $\hat{V}$  is oscillating between 0 and  $-1$ , which is the same as  $V_s$ , is actually much more crucial. If this is not guaranteed, then  $\hat{V}$  and  $V_s$  are not macroscopically indistinguishable in general. See Theorem 3.2 for this interesting observation.

**1.2.3. Viscous Case.** We may also consider the same inverse problem for the viscous Hamilton-Jacobi equation. For each  $p \in \mathbb{R}^n$ , the cell problem of interest is

$$(1.13) \quad -d\Delta w + H(p + Dw) + V(x) = \overline{H}_d(p) \quad \text{in } \mathbb{T}^n$$

for some given  $d > 0$ . Due to the presence of the diffusion term, any detailed analysis of the viscous effective Hamiltonian  $\overline{H}_d(p)$  becomes considerably more difficult even in one dimensional space. In this paper, we establish the following theorems which is a viscous analogue of Theorem 1.1.

**Theorem 1.5.** *Assume  $V \in C^\infty(\mathbb{T}^n)$ .*

(1) *Suppose that  $H \in C^2(\mathbb{R}^n)$ ,  $\sup_{\mathbb{R}^n} \|D^2 H\| < +\infty$  and  $H$  is superlinear. Then*

$$\overline{H}_d(p) \equiv H(p) \quad \Rightarrow \quad V \equiv 0.$$

(2) *Suppose that  $H(p) = |p|^2$ . If  $\overline{H}_d(p) = |p|^2 + o(|p|^2)$  for  $p$  in a neighborhood of the origin  $O \in \mathbb{R}^n$ , then  $V \equiv 0$ .*

When  $n = 1$  and  $H(p) = |p|^2$ , the inverse problem is actually equivalent to the inverse problem associated with the spectrum of the Hill operator  $L = -\frac{d^2}{dx^2} - V$ , which has been extensively studied in the literature. See the discussion in Section 4 for details.

**Theorem 1.6.** *Assume that  $V_1, V_2 \in C^\infty(\mathbb{T}^n)$ .*

(1) *Suppose that  $H \in C^2(\mathbb{R}^n)$ ,  $\sup_{\mathbb{R}^n} \|D^2 H\| < +\infty$  and  $H$  is superlinear. Then for  $i = 1, 2$  and any vector  $Q \in \mathbb{R}^n$  satisfying a Diophantine condition*

$$(1.14) \quad \int_{\mathbb{T}^n} V_i dx = \lim_{\lambda \rightarrow +\infty} (\overline{H}_i(\lambda P_\lambda) - H(\lambda P_\lambda)).$$

*Here  $P_\lambda \in \mathbb{R}^n$  is chosen such that  $DH(\lambda P_\lambda) = \lambda Q$ . In particular,*

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Rightarrow \quad \int_{\mathbb{T}^n} V_1 dx = \int_{\mathbb{T}^n} V_2 dx.$$

(2) *Let  $H(p) = |p|^2$ . If Fourier coefficients of  $V_i$  satisfy (1.8), then*

$$\overline{H}_{d1} \equiv \overline{H}_{d2} \quad \Rightarrow \quad \int_{\mathbb{T}^n} |V_1|^2 dx = \int_{\mathbb{T}^n} |V_2|^2 dx.$$

(3) *If  $n = 1$  and  $H(p) = |p|^2$ , then*

$$\overline{H}_{d1} \equiv \overline{H}_{d2} \quad \Rightarrow \quad \begin{cases} \int_{\mathbb{T}^1} V_1 dx = \int_{\mathbb{T}^1} V_2 dx \\ \int_{\mathbb{T}^1} |V_1|^2 dx = \int_{\mathbb{T}^1} |V_2|^2 dx. \end{cases}$$

**Remark 1.3.** *Part (3) in the above theorem is optimal. We cannot expect to get other identical norms of  $V_1$  and  $V_2$ . See Remark 4.1 for the connection with the KdV equation. This is different from the inviscid case.*

**Outline of the paper.** In Section 2, we provide the proofs of Theorems 1.1 and 1.2. Section 3 is devoted to the proofs of Theorems 1.3 and 1.4 and further detailed analysis in the one dimensional setting. The connection between the viscous case and the Hill operator will be discussed in Section 4. The proofs of Theorem 1.5 and 1.6 are also given there.

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## 2. SOME RESULTS IN THE GENERAL DIMENSIONAL CASE

**Proof of Theorem 1.1.** We first recall that as  $V_2 \equiv 0$ , we have  $\overline{H}_2 = H$  and therefore  $\overline{H}_1 = H$ .

Since  $\min_{\mathbb{R}^n} \overline{H}_1 = \min_{\mathbb{R}^n} H + \max_{\mathbb{R}^n} V_1$ , we deduce that  $\max_{\mathbb{R}^n} V_1 = 0$ . If  $V_1$  is not constantly zero, without loss of generality, we may assume that for some  $r > 0$

$$(2.15) \quad V_1(x) < 0 \quad \text{for } x \in B(0, r).$$

Since  $Q = DH(p_0)$  is an irrational vector, there exists  $T > 0$  such that for any  $x \in [0, 1]^n$ , there exists  $t_x \in [0, T]$ ,  $z_x \in \mathbb{Z}^n$  such that

$$(2.16) \quad |x - t_x Q - z_x| \leq \frac{r}{2}.$$

Let  $u(x, t)$  be the viscosity solution to

$$(2.17) \quad \begin{cases} u_t + H(Du) + V_1(x) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(x, 0) = p_0 \cdot x & \text{on } \mathbb{R}^n. \end{cases}$$

Then  $x \mapsto u(x, t) - p_0 \cdot x$  is  $\mathbb{Z}^n$ -periodic for each  $t \geq 0$ . Owing to (2.15), (2.16) and the following Lemma 2.1, we have that

$$u(x, T) > p_0 \cdot x - H(p_0)T \quad \text{for all } x \in \mathbb{R}^n.$$

The continuity and periodicity of  $x \mapsto u(x, T) - p_0 \cdot x$  then yield that

$$(2.18) \quad \delta = \min_{x \in \mathbb{R}^n} \{u(x, T) - p_0 \cdot x + H(p_0)T\} > 0.$$

Denote

$$u_1(x, t) = u(x, t + T) + H(p_0)T - \delta \quad \text{for } (x, t) \in \mathbb{R}^n \times [0, +\infty).$$

In light of (2.18),  $u_1(x, 0) \geq p_0 \cdot x = u(x, 0)$  for all  $x \in \mathbb{R}^n$ . The usual comparison principle implies that  $u_1 \geq u$ . Hence

$$\min_{x \in \mathbb{R}^n} \{u_1(x, T) - p_0 \cdot x + H(p_0)T\} \geq \delta.$$

Now for  $m \geq 2$ , define

$$u_m(x, t) = u_{m-1}(x, t + T) + H(p_0)T - \delta \quad \text{for } (x, t) \in \mathbb{R}^n \times [0, +\infty).$$



By using a similar argument and induction, we deduce that  $u_m \geq u$  and

$$\min_{x \in \mathbb{R}^n} \{u_m(x, T) - p_0 \cdot x + H(p_0)T\} \geq \delta \quad \text{for all } m \in \mathbb{N}.$$

Accordingly, for all  $x \in \mathbb{R}^n$  and  $m \in \mathbb{N}$ ,

$$u(x, mT) \geq p_0 \cdot x - H(p_0)mT + m\delta.$$

Therefore,

$$H(p_0) = \overline{H}_1(p_0) = \lim_{m \rightarrow \infty} -\frac{u(0, mT)}{mT} \leq H(p_0) - \frac{\delta}{T},$$

which is absurd. So  $V_1 \equiv 0$ .  $\square$

**Lemma 2.1.** *Suppose that  $V_1 \leq 0$  and  $u(x, t)$  is the viscosity solution of (2.17), which is*

$$\begin{cases} u_t + H(Du) + V_1(x) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(x, 0) = p_0 \cdot x & \text{on } \mathbb{R}^n. \end{cases}$$

*Suppose that  $H$  is differentiable at  $p_0$  and there exists a point  $(x_0, t_0) \in \mathbb{R}^n \times (0, +\infty)$  such that*

$$u(x_0, t_0) = p_0 \cdot x_0 - H(p_0)t_0.$$

*Then*

$$V_1(x_0 - (t_0 - s)DH(p_0)) = 0 \quad \text{for all } s \in [0, t_0].$$

*Proof.* Choose a convex and superlinear Hamiltonian  $\tilde{H} : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\tilde{H} \geq H$ ,  $\tilde{H}$  is differentiable at  $p_0$ ,

$$\tilde{H}(p_0) = H(p_0) \quad \text{and} \quad D\tilde{H}(p_0) = DH(p_0).$$

Let  $\tilde{u} : \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}$  be the unique solution to

$$\begin{cases} \tilde{u}_t + \tilde{H}(D\tilde{u}) + V_1(x) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ \tilde{u}(x, 0) = p_0 \cdot x & \text{on } \mathbb{R}^n. \end{cases}$$

By the comparison principle, we have that

$$u \geq \tilde{u} \geq p_0 \cdot x - \tilde{H}(p_0)t = p_0 \cdot x - H(p_0)t \quad \text{in } \mathbb{R}^n \times (0, +\infty).$$

Hence

$$(2.19) \quad \tilde{u}(x_0, t_0) = p_0 \cdot x_0 - \tilde{H}(p_0)t_0.$$

The optimal control formula gives that

$$\tilde{u}(x_0, t_0) = \min_{\gamma \in \Gamma_{x_0, t_0}} \left\{ p_0 \cdot \gamma(0) + \int_0^{t_0} (L(\dot{\gamma}(s)) - V_1(\gamma(s))) ds \right\}.$$

Here  $\Gamma_{x_0, t_0}$  is the collection of all absolutely continuous curves  $\gamma$  such that  $\gamma(t_0) = x_0$ . Assume that  $\tilde{u}(x_0, t_0) = p_0 \cdot \xi(0) + \int_0^{t_0} L(\dot{\xi}(s)) - V_1(\xi(s)) ds$  for some  $\xi \in \Gamma_{x_0, t_0}$ . Since  $L(\dot{\xi}(s)) + \tilde{H}(p_0) \geq p_0 \cdot \dot{\xi}(s)$  for a.e.  $s \in [0, t_0]$ , we have that

$$\begin{aligned} p_0 \cdot \xi(0) + \int_0^{t_0} L(\dot{\xi}(s)) - V_1(\xi(s)) ds &\geq p_0 \cdot x_0 - \tilde{H}(p_0)t_0 - \int_0^{t_0} V_1(\xi(s)) ds \\ &\geq p_0 \cdot x_0 - \tilde{H}(p_0)t_0 = p_0 \cdot x_0 - H(p_0)t_0. \end{aligned}$$

Accordingly,  $L(\dot{\xi}(s)) + \tilde{H}(p_0) = p_0 \cdot \dot{\xi}(s)$  for a.e.  $s \in [0, t_0]$  and  $V_1(\xi(s)) = 0$  for  $s \in [0, t_0]$ . So  $\dot{\xi}(s) = D\tilde{H}(p_0) = DH(p_0)$  for a.e.  $s \in [0, t_0]$ . Thus

$$V_1(\xi(s)) = V_1(x_0 - (t_0 - s)DH(p_0)) = 0 \quad \text{for all } s \in [0, t_0].$$

□

**Remark 2.1.** *If the set*

$$G = \{DH(p) : H \text{ is differentiable at } p \text{ for } p \in \mathbb{R}^n\}$$

*only contains rational vectors, the conclusion of Theorem 1.1 might fail. Below is a simple example.*

*Let  $n = 2$ . Suppose that  $V \in C^\infty(\mathbb{T}^2)$  and  $V \leq 0$ . Denote  $Q = [0, 1]^2$ . We can think of  $V$  as a function defined on  $Q$  with periodic boundary condition. Assume further that  $\partial Q \subset \{V = 0\}$ . Let*

$$H(p) = \max\{K_1(p_1), K_2(p_2)\} \quad \text{for all } p = (p_1, p_2) \in \mathbb{R}^2.$$

*Here  $K_i \in C(\mathbb{R})$  is coercive for  $i = 1, 2$ . Then it is not hard to verify that*

$$\overline{H}(p) = H(p) = \max\{K_1(p_1), K_2(p_2)\} \quad \text{for all } p \in \mathbb{R}^2.$$

**Proof of Theorem 1.2 (Part 1).** We first prove (1.5) for  $i = 1$ .

Since  $Q$  satisfies a Diophantine condition, there exists a unique smooth periodic solution  $v$  (up to an additive constant) to

$$Q \cdot Dv = a_1 - V_1 \quad \text{in } \mathbb{T}^n,$$

for  $a_1 = \int_{\mathbb{T}^n} V_1 dx$ . Then it is easy to see that for  $v_\lambda = \frac{v}{\lambda}$ ,

$$H(\lambda P_\lambda + Dv_\lambda) + V(x) = H(\lambda P_\lambda) + a_1 + O\left(\frac{1}{\lambda^2}\right) \quad \text{in } \mathbb{T}^n.$$

Let  $w_\lambda \in C(\mathbb{T}^n)$  be a viscosity solution to

$$H(\lambda P_\lambda + Dw_\lambda) + V(x) = \overline{H}_1(\lambda P_\lambda) \quad \text{in } \mathbb{R}^n.$$

By looking at the places where  $w_\lambda - v_\lambda$  attains its maximum and minimum, we get that

$$\overline{H}_1(\lambda P_\lambda) = H(\lambda P_\lambda) + a_1 + O\left(\frac{1}{\lambda^2}\right),$$

which yields (1.5). □

**Lemma 2.2.** *Assume that  $V \in C^\infty(\mathbb{T}^n)$ . Let  $\overline{H}$  be the effective Hamiltonian associated with  $\frac{1}{2}|p|^2 + V$ . Then*

$$|p|^2 \geq \max_{\{Q \in \partial \overline{H}(p)\}} p \cdot Q.$$

*Proof.* It suffices to verify the above inequality at  $p_0 \in \mathbb{R}^n$ , which is a differentiable point of  $\overline{H}$ . Let  $w \in C^{0,1}(\mathbb{T}^n)$  be a viscosity solution to the cell problem

$$\frac{1}{2}|p_0 + Dw|^2 + V(x) = \overline{H}(p_0) \quad \text{in } \mathbb{T}^n.$$

Choose  $\mu$  to be a Mather measure associated with the Hamiltonian  $\frac{1}{2}|p_0 + p|^2 + V$ . Mather measures are probability Borel measures on  $\mathbb{R}^n \times \mathbb{T}^n = \{(q, x) \mid q \in \mathbb{R}^n, x \in \mathbb{T}^n\}$  which minimize the Lagrangian action among Euler-Lagrange flow invariant probability Borel measures. See [10, 12] for the precise definition and relevant properties. Denote by  $\sigma$  the projection of  $\mu$  to the base space  $\mathbb{T}^n$ . Then  $w$  is  $C^{1,1}$  on  $\text{spt}(\sigma)$ ,

$$\frac{1}{2}|p_0 + Dw|^2 + V(x) = \overline{H}(p_0) \quad \text{on } \text{spt}(\sigma).$$

Moreover,

$$p_0 + Dw(x) = q \quad \text{on } \text{spt}(\mu), \quad \int_{\mathbb{R}^n \times \mathbb{T}^n} q \, d\mu = D\overline{H}(p_0),$$

and

$$\int_{\mathbb{R}^n \times \mathbb{T}^n} q \cdot D\phi(x) \, d\mu = 0 \quad \text{for any } \phi \in C^1(\mathbb{T}^n).$$

Therefore

$$\int_{\mathbb{R}^n \times \mathbb{T}^n} |q|^2 \, d\mu = p_0 \cdot D\overline{H}(p_0)$$

and

$$|p_0|^2 = \int_{\mathbb{R}^n \times \mathbb{T}^n} |q - Dw|^2 \, d\mu = p_0 \cdot D\overline{H}(p_0) + \int_{\mathbb{R}^n \times \mathbb{T}^n} |Dw|^2 \, d\mu.$$

□

**Proof of Theorem 1.2 (Part 2).** Next we prove (1.6). Since  $Q$  is just irrational, the asymptotic expansion method is no longer applicable. The proof becomes more involved and relies on the special structure of the quadratic Hamiltonian.

**Step 1:** We claim that for any  $\lambda > 0$ , there exists  $\tilde{Q}_\lambda \in \partial\overline{H}_1(\lambda Q)$  such that

$$(2.20) \quad \lim_{\lambda \rightarrow +\infty} \left( \overline{H}_1(\lambda Q) - \frac{1}{2}\tilde{Q}_\lambda \cdot \lambda Q \right) = \int_{\mathbb{T}^n} V_1 \, dx.$$

It suffices to prove the above claim for any sequence  $\{\lambda_m\}$  converging to  $+\infty$ . Without loss of generality, we consider the sequence  $\{\lambda_m\}$  such that  $\lambda_m = m$  for all  $m \in \mathbb{N}$ . For  $m \geq 1$ , let

$$H_m(p, x) = \frac{1}{2}|Q + p|^2 + \frac{1}{m^2}V_1(x) \quad \text{for all } (p, x) \in \mathbb{R}^n \times \mathbb{R}^n,$$

and denote by  $\overline{H}_m$  its corresponding effective Hamiltonian. Let  $w_m \in C(\mathbb{T}^n)$  be a solution to the following cell problem

$$(2.21) \quad \frac{1}{2}|Q + Dw_m|^2 + \frac{1}{m^2}V_1(x) = \overline{H}_m(Q) \quad \text{in } \mathbb{T}^n.$$

By a simple scaling argument, we can easily check that

$$\overline{H}_m(Q) = \frac{1}{m^2}\overline{H}_1(mQ).$$

Choose  $\mu_m$  to be a Mather measure associated with  $H_m$ . Denote  $\sigma_m$  as the projection of  $\mu_m$  to the base space  $\mathbb{T}^n$ . Then  $w_m$  is  $C^{1,1}$  on  $\text{spt}(\sigma_m)$ ,

$$\frac{1}{2}|Q + Dw_m|^2 + \frac{1}{m^2}V_1(x) = \overline{H}_m(Q) \quad \text{on } \text{spt}(\sigma_m).$$

Moreover,

$$Q + Dw_m(x) = q \quad \text{on } \text{spt}(\mu_m), \quad \int_{\mathbb{R}^n \times \mathbb{T}^n} q \, d\mu_m = Q_m \in \partial \overline{H}_m(Q)$$

and

$$\int_{\mathbb{R}^n \times \mathbb{T}^n} q \cdot D\phi(x) \, d\mu_m = 0 \quad \text{for any } \phi \in C^1(\mathbb{T}^n).$$

Accordingly,

$$\overline{H}_m(Q) = \int_{\mathbb{R}^n \times \mathbb{T}^n} \frac{1}{2}|q|^2 + \frac{1}{m^2}V_1(x) \, d\mu_m \quad \text{and} \quad \int_{\mathbb{R}^n \times \mathbb{T}^n} \frac{1}{2}|q|^2 \, d\mu_m = \frac{1}{2}Q \cdot Q_m.$$

Therefore

$$\int_{\mathbb{R}^n \times \mathbb{T}^n} V_1(x) \, d\mu_m = m^2 \left( \overline{H}_m(Q) - \frac{1}{2}Q \cdot Q_m \right) = \overline{H}_1(mQ) - \frac{1}{2}mQ \cdot mQ_m.$$

Upon passing a subsequence if necessary, we may assume that

$$\mu_m \rightharpoonup \mu \quad \text{weakly in } \mathbb{R}^n \times \mathbb{T}^n,$$

for some probability measure  $\mu$  in  $\mathbb{R}^n \times \mathbb{T}^n$ . Let  $\sigma$  be the projection of  $\mu$  to the base space  $\mathbb{T}^n$ . Owing to the following Lemma 2.3, we have that

$$Q = q \quad \text{on } \text{spt}(\mu) \quad \text{and} \quad \int_{\mathbb{T}^n} Q \cdot D\phi(x) \, d\sigma = 0 \quad \text{for any } \phi \in C^1(\mathbb{T}^n).$$

Hence

$$\int_{\mathbb{T}^n} \phi(x) \, d\sigma = \frac{1}{T} \int_{\mathbb{T}^n} \int_0^T \phi(x + tQ) \, dt d\sigma.$$

Sending  $T \rightarrow +\infty$ , since  $Q$  is a irrational vector, we have that  $\sigma$  is actually the Lebesgue measure, i.e.,

$$\int_{\mathbb{T}^n} \phi(x) \, d\sigma = \int_{\mathbb{T}^n} \phi(x) \, dx \quad \text{for any } \phi \in C^1(\mathbb{T}^n).$$

So

$$\int_{\mathbb{T}^n} V_1(x) \, dx = \lim_{m \rightarrow +\infty} \int_{\mathbb{R}^n \times \mathbb{T}^n} V_1(x) \, d\mu_m = \lim_{m \rightarrow +\infty} \left( \overline{H}_1(mQ) - \frac{1}{2}mQ \cdot mQ_m \right).$$

Since  $\tilde{Q}_m = mQ_m \in \partial \overline{H}_1(mQ)$ , our claim (2.20) holds.

**Step 2:** Write

$$f(\lambda) = \frac{1}{2}\lambda^2|Q|^2 - \overline{H}_1(\lambda Q) \quad \text{for } \lambda \in (0, \infty).$$

Then  $f$  is locally Lipschitz continuous. Owing to Lemma 2.2,  $f' \geq 0$  a.e. Also  $f$  is uniformly bounded since

$$0 \leq \frac{1}{2}|p|^2 - \overline{H}_1(p) \leq - \int_{\mathbb{T}^n} V_1 \, dx.$$

Hence  $\lim_{\lambda \rightarrow +\infty} f(\lambda)$  exists and  $\liminf_{\lambda \rightarrow +\infty} \lambda f'(\lambda) = 0$ . So there exists a sequence  $\lambda_m \rightarrow +\infty$  as  $m \rightarrow +\infty$  such that  $f$  is differentiable at  $\lambda_m$  and

$$\lim_{m \rightarrow +\infty} \lambda_m f'(\lambda_m) = 0.$$

Note that if  $f$  is differentiable at  $\lambda$ , then

$$f'(\lambda) = \lambda|Q|^2 - Q \cdot q$$

for any  $q \in \partial \overline{H}_1(\lambda Q)$ . Accordingly,

$$f'(\lambda_m) = \lambda_m|Q|^2 - Q \cdot \tilde{Q}_{\lambda_m}$$

for  $\tilde{Q}_{\lambda_m} \in \partial \overline{H}_1(\lambda_m Q)$  from (2.20). Thus

$$\lim_{m \rightarrow +\infty} (\lambda_m^2|Q|^2 - \lambda_m Q \cdot \tilde{Q}_{\lambda_m}) = 0.$$

Together with (2.20), we have that

$$\lim_{\lambda \rightarrow +\infty} f(\lambda) = \lim_{m \rightarrow +\infty} f(\lambda_m) = - \int_{\mathbb{T}^n} V_1 dx.$$

Combining this with Lemma 2.2 and (2.20), we also obtain (1.7).  $\square$

**Lemma 2.3.** *Assume  $V \in C^\infty(\mathbb{T}^n)$ . For  $\varepsilon > 0$ , let  $v_\varepsilon \in C(\mathbb{T}^n)$  be a viscosity solution to*

$$\frac{1}{2}|P + Dv_\varepsilon|^2 + \varepsilon V(x) = \overline{H}_\varepsilon(P) \quad \text{in } \mathbb{T}^n.$$

*Then*

$$(2.22) \quad \lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathcal{R}_\varepsilon} |Dv_\varepsilon(x)| = 0.$$

*Here  $\mathcal{R}_\varepsilon = \{x \in \mathbb{T}^n : v_\varepsilon \text{ is differentiable at } x\}$ .*

*Proof.* We argue by contradiction. If (2.22) were false, then there would exist  $\delta > 0$  such that, by passing to a subsequence if necessary,

$$\lim_{\varepsilon \rightarrow 0} |Dv_\varepsilon(x_\varepsilon)| \geq \delta$$

and  $v_\varepsilon(x_\varepsilon) = 0$  for some  $x_\varepsilon \in \mathcal{R}_\varepsilon \cap [0, 1]^n$ . Let  $\xi_\varepsilon : (-\infty, 0] \rightarrow \mathbb{R}^n$  be the backward characteristic associated with  $P \cdot x + v_\varepsilon$  with  $\xi_\varepsilon(0) = x_\varepsilon$ . Then

$$\ddot{\xi}_\varepsilon = -\varepsilon DV(\xi_\varepsilon) \quad \text{and} \quad \dot{\xi}_\varepsilon = P + Dv_\varepsilon(\xi_\varepsilon),$$

and for any  $t_2 < t_1 \leq 0$ ,

$$(2.23) \quad P \cdot \xi_\varepsilon(t_1) + v_\varepsilon(\xi_\varepsilon(t_1)) - P \cdot \xi_\varepsilon(t_2) - v_\varepsilon(\xi_\varepsilon(t_2)) = \int_{t_2}^{t_1} \frac{1}{2} |\dot{\xi}_\varepsilon(s)|^2 + \overline{H}_\varepsilon(P) ds.$$

Upon a subsequence if necessary, we assume that

$$\lim_{\varepsilon \rightarrow 0} \xi_\varepsilon = \xi_0 \quad \text{uniformly in } C^1[-1, 0].$$

It is clear that  $\dot{\xi}_0 \equiv \tilde{P} = P + P_1$  for some  $|P_1| \geq \delta$ . However, it is easy to see that

$$\lim_{\varepsilon \rightarrow 0} \overline{H}_\varepsilon(P) = \frac{1}{2}|P|^2 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} v_\varepsilon = 0 \quad \text{uniformly in } \mathbb{R}^n.$$

Owing to (2.23), we obtain that

$$P \cdot \tilde{P} \geq \frac{1}{2}|\tilde{P}|^2 + \frac{1}{2}|P|^2.$$

So  $P = \tilde{P}$ . This is a contradiction.  $\square$

**Remark 2.2.** If  $v_\varepsilon \in C^\infty(\mathbb{T}^n)$  (e.g. in the regime of classical KAM theory), standard maximum principle arguments lead to

$$\max_{\mathbb{T}^n} |D^2 v_\varepsilon| \leq C\sqrt{\varepsilon}.$$

for a constant  $C$  depending only on  $V$  and the dimension  $n$ . Hence

$$\max_{\mathbb{T}^n} |Dv_\varepsilon| = O(\sqrt{\varepsilon}).$$

If  $P$  is an irrational vector, (1.6) implies that  $\int_{\mathbb{T}^n} |Dv_\varepsilon|^2 dx = o(\varepsilon)$ . Accordingly,

$$\max_{\mathbb{T}^n} |Dv_\varepsilon| = o(\sqrt{\varepsilon}).$$

Higher order approximations for more general Hamiltonian can be found in [8].

**Proof of Theorem 1.2 (Part 3).** We now show that

$$\int_{\mathbb{T}^n} |V_1|^2 dx = \int_{\mathbb{T}^n} |V_2|^2 dx.$$

under the decay assumption (1.8).

Let  $Q$  be a vector satisfying a Diophantine condition. For  $i = 1, 2$ , we can explicitly solve the following two equations in  $\mathbb{T}^n$  by computing Fourier coefficients

$$(2.24) \quad \begin{cases} Q \cdot Dv_{i1} = a_{i1} - V_i, \\ Q \cdot Dv_{i2} = a_{i2} - \frac{1}{2}|Dv_{i1}|^2. \end{cases}$$

Here  $a_{i1} = \int_{\mathbb{T}^n} V_i dx$  and  $a_{i2} = \frac{1}{2} \int_{\mathbb{T}^n} |Dv_{i1}|^2 dx$ . Then for  $\varepsilon > 0$  and  $i = 1, 2$ ,  $v_{i\varepsilon} = \varepsilon v_{i1} + \varepsilon^2 v_{i2}$  satisfy

$$\frac{1}{2}|Q + Dv_{i\varepsilon}|^2 + \varepsilon V_i = \frac{1}{2}|Q|^2 + \varepsilon a_{i1} + \varepsilon^2 a_{i2} + O(\varepsilon^3).$$

Suppose that  $w_{i\varepsilon} \in C(\mathbb{T}^n)$  is a viscosity solution to

$$\frac{1}{2}|Q + Dw_{i\varepsilon}|^2 + \varepsilon V_i = \overline{H}_{i\varepsilon}(Q) \quad \text{in } \mathbb{T}^n.$$

Here  $\overline{H}_{i\varepsilon}$  is the effective Hamiltonian associated with  $\frac{1}{2}|p|^2 + \varepsilon V_i$ . Since  $v_{i\varepsilon} \in C^\infty(\mathbb{T}^n)$ , by looking at places where  $v_{i\varepsilon} - w_{i\varepsilon}$  attains its maximum and minimum, we derive that

$$\overline{H}_{i\varepsilon}(Q) = \frac{1}{2}|Q|^2 + \varepsilon a_{i1} + \varepsilon^2 a_{i2} + O(\varepsilon^3).$$

Note that, for  $i = 1, 2$ ,

$$\overline{H}_{i\varepsilon}(Q) = \varepsilon \overline{H}_i \left( \frac{Q}{\sqrt{\varepsilon}} \right).$$

Accordingly,

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Rightarrow \quad \begin{cases} \int_{\mathbb{T}^n} V_1 dx = \int_{\mathbb{T}^n} V_2 dx \\ \int_{\mathbb{T}^n} |Dv_{11}|^2 dx = \int_{\mathbb{T}^n} |Dv_{21}|^2 dx. \end{cases}$$

Recall that  $\{\lambda_{k1}\}_{k \in \mathbb{Z}^n}$  and  $\{\lambda_{k2}\}_{k \in \mathbb{Z}^n}$  are the Fourier coefficients of  $V_1$  and  $V_2$  respectively. Since  $V_1, V_2 \in C^\infty(\mathbb{T}^n)$ ,  $\lambda_{ki}$  decays faster than any power of  $k$ , i.e. for  $i = 1, 2$  and any  $m \in \mathbb{N}$ ,

$$(2.25) \quad \lim_{|k| \rightarrow +\infty} |\lambda_{ki}| \cdot |k|^m = 0.$$

Note that we do not need the strong decay condition (1.8) at this point. Then for any  $Q$  satisfying a Diophantine condition,  $\int_{\mathbb{T}^n} |Dv_{11}|^2 dx = \int_{\mathbb{T}^n} |Dv_{21}|^2 dx$  implies that

$$(2.26) \quad \sum_{0 \neq k \in \mathbb{Z}^n} \frac{|k|^2 |\lambda_{k1}|^2}{|Q \cdot k|^2} = \sum_{0 \neq k \in \mathbb{Z}^n} \frac{|k|^2 |\lambda_{k2}|^2}{|Q \cdot k|^2}.$$

Our goal is to verify that

$$\sum_{0 \neq k \in \mathbb{Z}^n} |\lambda_{k1}|^2 = \sum_{0 \neq k \in \mathbb{Z}^n} |\lambda_{k2}|^2.$$

For  $\delta > 0$  and  $\tau' = \frac{\tau}{2}$ , denote

$$\Omega_\delta = \left\{ Q \in B_1(0) : |Q \cdot k| \geq \frac{\delta}{|k|^{n-1+\tau'}} \text{ for all } 0 \neq k \in \mathbb{Z}^n \right\}.$$

Clearly,  $\Omega_\delta$  is decreasing with respect to  $\delta$  and  $\cup_{\delta > 0} \Omega_\delta$  has full measure, i.e.,  $|\cup_{\delta > 0} \Omega_\delta| = |B_1(0)|$ . Due to (2.25), we can take derivatives of the above equality (2.26). It leads to that, for any  $m \in \mathbb{N}$ ,  $\delta > 0$ ,

$$(2.27) \quad \sum_{0 \neq k \in \mathbb{Z}^n} \frac{|k|^{2m} |\lambda_{k1}|^2}{|Q \cdot k|^{2m}} = \sum_{0 \neq k \in \mathbb{Z}^n} \frac{|k|^{2m} |\lambda_{k2}|^2}{|Q \cdot k|^{2m}} \text{ for a.e. } Q \in \Omega_\delta.$$

Owing to (1.8), we have that

$$\sum_{0 \neq k \in \mathbb{Z}^n} (|\lambda_{k2}|^2 + |\lambda_{k2}|^2) e^{\frac{|k|}{|Q \cdot k|}} < +\infty \text{ for any } Q \in \Omega_\delta.$$

This, together with (2.27), implies that

$$\sum_{0 \neq k \in \mathbb{Z}^n} d_k \left( 1 - \cos \left( \frac{|k|}{|Q \cdot k|} \right) \right) = 0 \text{ for a.e. } Q \in \Omega_\delta.$$

for  $d_k = |\lambda_{k2}|^2 - |\lambda_{k1}|^2$ . Sending  $\delta \rightarrow 0$ , we derive that

$$\sum_{0 \neq k \in \mathbb{Z}^n} d_k \left( 1 - \cos \left( \frac{|k|}{|Q \cdot k|} \right) \right) = 0 \text{ for a.e. } Q \in B_1(0).$$

Taking integration of the above with respect to  $Q$  in  $B_1(0)$  leads to

$$\sum_{0 \neq k \in \mathbb{Z}^n} d_k = 0.$$

□

**Remark 2.3.** From the above proof, we actually derive that for  $i = 1, 2$ ,

$$(2.28) \quad \int_{\mathbb{T}^n} |V_i|^2 dx = \frac{2}{c_0} \int_{B_1(0)} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\Delta^{(m-1)} a_{i2}(Q)}{(2m)!(2m-1)!} dQ.$$

Here  $c_0 = \int_{B_1(0)} \left(1 - \cos \frac{1}{|x_1|}\right) dx$ ,  $\Delta^{(m)}$  represents the  $m$ -th Laplacian and  $a_{i2}(Q) = \frac{1}{2} \int_{\mathbb{T}^n} |Dv_{i1}|^2 dx$ , i.e., the coefficient of  $\varepsilon^2$  in the asymptotic expansion. Furthermore, it is clear that the conclusion of Theorem 1.2 only depends on the behavior of the effective Hamiltonian when  $|p|$  is large. In fact, when  $n \geq 2$ , for any  $M > 0$ , it is easy to construct two different smooth periodic functions  $V_1$  and  $V_2$  with big bumps such that  $V_1 \geq V_2$ ,  $\max_{\mathbb{T}^n} V_1 = \max_{\mathbb{T}^n} V_2 = 0$  and

$$\overline{H}_1(p) \equiv \overline{H}_2(p) \quad \text{when } \overline{H}_2(p) \leq M.$$

See (9.7) in [4] for instance.

### 3. DETAILED ANALYSIS IN THE ONE DIMENSIONAL CASE

**3.1. Convex Case.** We first present a proof of Theorem 1.3.

**Proof of Theorem 1.3.** The proof of part (1) is quite straightforward. By the coercivity of  $H$  and the stability of  $\overline{H}$ , we may assume that  $H(0) = 0 = \min_{\mathbb{R}} H$ ,  $H$  is strictly increasing on  $[0, +\infty)$  and is strictly decreasing on  $(-\infty, 0]$ . In light of this assumption, the formula of the effective Hamiltonian is given by

$$\begin{cases} p = \int_0^1 H_+^{-1}(\overline{H}_i(p) - V_i(x)) dx & \text{for } p \geq p_+, \\ \overline{H}(p) \equiv 0 & \text{for } p \in [p_-, p_+], \\ p = \int_0^1 H_-^{-1}(\overline{H}_i(p) - V_i(x)) dx & \text{for } p \leq p_-. \end{cases}$$

Here

$$p_{\pm} = \int_0^1 H_{\pm}^{-1}(-V_1(x)) dx = \int_0^1 H_{\pm}^{-1}(-V_2(x)) dx,$$

and  $H_-^{-1}$ ,  $H_+^{-1}$  are the inverses of  $H$  on  $[0, +\infty)$  and  $(-\infty, 0]$  respectively. Clearly,  $\overline{H}_1 \equiv \overline{H}_2$ .

Let us prove the second part (2). Since  $\min \overline{H}_i = \max_{\mathbb{R}} V_i$ , we may assume that  $\max_{\mathbb{R}} V_i = 0$  for  $i = 1, 2$ . Apparently, for  $i = 1, 2$  and  $c \geq H(a)$ ,

$$\max \{p \in \mathbb{R} : \overline{H}_i(p) = c\} = \int_0^1 \psi(c - V_i(x)) dx.$$

Hence

$$\int_0^1 \psi(\lambda - V_1(x)) dx = \int_0^1 \psi(\lambda - V_2(x)) dx \quad \text{for all } \lambda \geq H(a).$$

Therefore

$$\int_{-M}^0 \psi(\lambda - t) dF_1(t) = \int_{-M}^0 \psi(\lambda - t) dF_2(t) \quad \text{for all } \lambda \geq H(a).$$



Here  $F_i(t) = |\{x \in [0, 1] : V_i(x) \leq t\}|$  is the distribution function of  $V_i$  for  $i = 1, 2$ , and  $-M < \min\{\min V_1, \min V_2\}$ . Denote  $G(t) = F_1(t) - F_2(t) \in BV[-M, 0]$ . Then

$$\int_{-M}^0 \psi(\lambda - t) dG(t) = 0,$$

and  $G(0) = G(-M) = 0$ . By integration by parts, we derive that

$$\int_{-M}^0 \psi'(\lambda - t) G dt = 0.$$

For  $m \in \mathbb{N} \cap [H(a) + 1, +\infty)$ , choose  $x_m \in I_m = [m, M + m]$  such that

$$|\psi'(x_m)| = \max_{x \in I_m} |\psi'(x)|.$$

Then

$$\lim_{m \rightarrow +\infty} \frac{\max_{x \in I_m} |\psi'(x) - \psi'(x_m)|}{|\psi'(x_m)|} \leq \lim_{m \rightarrow +\infty} \max_{x \in I_m} \frac{M |\psi''(x)|}{|\psi'(x)|} = 0.$$

Hence

$$(3.29) \quad \int_{-M}^0 G(t) dt = \lim_{m \rightarrow +\infty} \frac{1}{M} \int_{-M}^0 \frac{\psi'(x_m) - \psi'(m - t)}{\psi'(x_m)} G(t) dt = 0.$$

Define  $G_0 = G$  and  $G_k(t) = \int_{-M}^t G_{k-1}(s) ds$  for all  $k \in \mathbb{N}$ . Through integration by parts, using (1.9) and the similar approach to obtain the above (3.29), we derive that for  $k \geq 1$ ,

$$\int_{-M}^0 \psi^{(k+1)}(\lambda - t) G_k(t) dt = 0 \quad \text{for all } \lambda \geq H(a) + 1,$$

and  $G_k(-M) = G_k(0) = 0$ , i.e.,

$$\int_{-M}^0 G_{k-1}(t) dt = 0.$$

Via integration by parts, it is easy to prove that the above equality leads to

$$\int_{-M}^0 t^k G dt = k! (-1)^k \int_{-M}^0 G_k dx = 0 \quad \text{for all } k \in \mathbb{N} \cup \{0\}.$$

Hence  $G = 0$  a.e. in  $[-M, 0]$ . Thus  $F_1(t) = F_2(t)$  a.e. in  $[-M, 0]$ . Since both  $F_1$  and  $F_2$  are right-hand continuous, we have that

$$F_1 \equiv F_2.$$

□

**Remark 3.1.** Clearly, the second part in the above theorem is not true if  $H$  is not strongly superlinear. For example, for  $H(p) = |p|$ , then for  $i = 1, 2$ ,

$$\overline{H}_i(p) = \begin{cases} 0 & \text{if } |p| \leq -\int_0^1 V_i(x) dx, \\ |p| + \int_0^1 V_i(x) dx & \text{if } |p| \geq -\int_0^1 V_i(x) dx. \end{cases}$$

Therefore, in this case,  $\overline{H}_1 = \overline{H}_2$  if and only if

$$\int_0^1 V_1(x) dx = \int_0^1 V_2(x) dx,$$

which is clearly much weaker than assertion of (2).

In fact, it is not even true for some strictly convex Hamiltonians. For example, let  $\psi \geq 0$  be a smooth and strictly concave function satisfying that  $\psi(0) = 0$  and

$$\psi(q) = \Psi(q) = |q|(1 - e^{-|q|}) \quad \text{for } |q| \geq 3.$$

Then  $H = \psi^{-1}$  is strictly convex and coercive. Choose two smooth periodic functions  $V_1$  and  $V_2$  with different distributions such that

- $\max_{\mathbb{R}} V_i = 0$  and  $\{V_i \geq -3\} = [\frac{1}{4}, \frac{3}{4}]$  for  $i = 1, 2$ ;
- $V_1 = V_2$  on  $[\frac{1}{4}, \frac{3}{4}]$  and  $\int_0^1 V_1(x) dx = \int_0^1 V_2(x) dx$ ;
- Furthermore,

$$\int_0^1 V_1(x) e^{V_1(x)} dx = \int_0^1 V_2(x) e^{V_2(x)} dx, \quad \int_0^1 e^{V_1(x)} dx = \int_0^1 e^{V_2(x)} dx.$$

Clearly, for any  $c \geq 0$ ,

$$\begin{aligned} \int_0^1 \psi(c - V_1(x)) dx - \int_0^1 \psi(c - V_2(x)) dx \\ = \int_0^1 \Psi(c - V_1(x)) dx - \int_0^1 \Psi(c - V_2(x)) dx = 0. \end{aligned}$$

Thus

$$\overline{H}_1 \equiv \overline{H}_2.$$

**Remark 3.2.** When  $n \geq 2$ , that  $V_1$  and  $V_2$  have the same distribution is not sufficient to yield that  $\overline{H}_1 \equiv \overline{H}_2$ . Here is a simple example for  $n = 2$ .

Let  $H(p) = |p|^2$ . For  $x = (x_1, x_2) \in \mathbb{R}^2$ , set

$$V_1(x) = -\sin(2\pi x_2) \quad \text{and} \quad V_2(x) = -\sin(2\pi x_1).$$

Clearly,  $V_1$  and  $V_2$  have the same distribution. However,  $\overline{H}_1 \neq \overline{H}_2$ . In fact, it is easy to see that

$$\overline{H}_1(p) = |p_1|^2 + h(p_2) \quad \text{and} \quad \overline{H}_2(p) = |p_2|^2 + h(p_1).$$

Here  $h(t)$  is given by

$$h(t) = \begin{cases} 1 & \text{if } |t| \leq \int_0^1 \sqrt{1 + \sin(2\pi\theta)} d\theta, \\ |t| = \int_0^1 \sqrt{h(t) + \sin(2\pi\theta)} d\theta & \text{otherwise.} \end{cases}$$

In fact, we should be able to identify precise necessary and sufficient conditions for  $\overline{H}_1 \equiv \overline{H}_2$  when  $V_1$  and  $V_2$  have finite frequencies. This will be in a forthcoming paper.

### 3.2. Nonconvex Case.

**Proof of Theorem 1.4.** We proceed the proof in several steps.

**Step 1:** For  $s \in (0, 1)$ , denote

$$p_{+,s} = \max\{p \geq 0 : \overline{H}_s(p) = 0\}.$$

Let

$$\begin{cases} \psi_1 = (F|_{[\theta_1, \infty)})^{-1} : [\frac{1}{3}, +\infty) \rightarrow [\theta_1, +\infty), \\ \psi_2 = (F|_{[\theta_2, \theta_1]})^{-1} : [\frac{1}{3}, \frac{1}{2}] \rightarrow [\theta_2, \theta_1], \\ \psi_3 = (F|_{[0, \theta_2]})^{-1} : [0, \frac{1}{2}] \rightarrow [0, \theta_2]. \end{cases}$$

Define  $f_s$  to be a periodic function satisfying that

$$f_s(x) = \begin{cases} \psi_3(-V_s(x)) & \text{for } x \in [0, \frac{s}{3}] \cup [\frac{1}{2} + \frac{s}{2}, 1] \\ \psi_1(-V_s(x)) & \text{for } x \in (\frac{s}{3}, \frac{1}{2} + \frac{s}{2}). \end{cases}$$

It is easy to check that any solution  $w' = f_s(x)$  is a viscosity solution to

$$F(|w'|) + V_s(x) = 0 \quad \text{in } \mathbb{R}.$$

We claim that

$$\begin{aligned} (3.30) \quad p_{+,s} &= \int_{[0,1]} f_s(x) dx \\ &= \int_{\frac{1}{3}}^1 \psi_1(y) dy + \int_0^{\frac{1}{3}} \psi_3(y) dy + (1-s) \int_{\frac{1}{3}}^{\frac{1}{2}} (\psi_3 - \psi_1)(y) dy. \end{aligned}$$

In fact, assume that  $v \in C^{0,1}(\mathbb{R})$  is a periodic viscosity solution to

$$F(|p_{+,s} + v'|) + V_s(x) = 0 \quad \text{in } \mathbb{R}.$$

Write  $u = p_{+,s}x + v$ . Obviously,

$$u'(x) \leq f_s(x) \quad \text{for a.e. } x \in [0, 1] \setminus \left[ \frac{1}{2} + \frac{s}{2}, \frac{2}{3} + \frac{s}{3} \right].$$

Suppose that there exists  $x_0 \in (\frac{1}{2} + \frac{s}{2}, \frac{2}{3} + \frac{s}{3})$  such that  $u'(x_0)$  exists and

$$u'(x_0) = \psi_j(-V_s(x_0)) \quad \text{for } j = 1 \text{ or } j = 2.$$

Since  $u'((\frac{2}{3} + \frac{s}{3}, 1)) \subset (-\infty, \theta_3]$ , due to Lemma 3.1, we have that

$$F(\theta_2) + V_s(x_1) \leq 0 \quad \text{for some } x_1 \in \left[ x_0, \frac{2}{3} + \frac{s}{3} \right].$$

This is impossible since  $-V_s < \frac{1}{2}$  in  $(\frac{1}{2} + \frac{s}{2}, \frac{2}{3} + \frac{s}{3})$ . Accordingly,

$$u'(x) \leq \psi_3(-V_s(x)) \quad \text{in } \left( \frac{1}{2} + \frac{s}{2}, \frac{2}{3} + \frac{s}{3} \right).$$

So  $u'(x) \leq f_s(x)$  and hence the first equality of claim (3.30) holds. An easy calculation leads to

$$\int_{[0,1]} f_s(x) dx = \int_{\frac{1}{3}}^1 \psi_1(y) dy + \int_0^{\frac{1}{3}} \psi_3(y) dy + (1-s) \int_{\frac{1}{3}}^{\frac{1}{2}} (\psi_3 - \psi_1)(y) dy,$$

which implies the second equality of (3.30).

**Step 2:** We show that  $\hat{V}$  and  $V_s$  are macroscopically indistinguishable if (1.11) holds. Now we only assume that  $H : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and coercive. Fix  $s \in (0, 1)$ ,  $1 \leq i \leq m-1$ , and define  $\tilde{V} \in C(\mathbb{T})$  as

$$\tilde{V}(x) = V_s \left( \frac{x - a_i}{a_{i+1} - a_i} \right) \quad \text{for } x \in [a_i, a_{i+1}].$$

Let  $\tilde{H}$  be the effective Hamiltonian associated with  $H(p) + \tilde{V}$ . Since  $\tilde{V}$  is basically a piecewise rescaling of  $V_s$ ,  $\tilde{H} \equiv \overline{H}_s$ . For fixed  $p \in \mathbb{R}$ , let  $v$  be a continuous periodic viscosity solution to

$$H(p + v') + V_s(x) = \overline{H}_s(p) \quad \text{in } \mathbb{R}$$

subject to  $v(0) = v(1) = 0$ . Define, for  $x \in [0, 1]$ ,

$$\tilde{v}(x) = (a_{i+1} - a_i)v \left( \frac{x - a_i}{a_{i+1} - a_i} \right) \quad \text{for } x \in [a_i, a_{i+1}], \quad 1 \leq i \leq m-1,$$

and extend  $\tilde{v}$  to  $\mathbb{R}$  in a periodic way. It is easy to check that  $\tilde{v}$  is a viscosity to

$$H(p + \tilde{v}') + \tilde{V}(x) = \overline{H}_s(p) \quad \text{in } \mathbb{R}.$$

Next we set  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  as follows: when  $x \in [0, 1]$ ,

$$\tau(x) = \begin{cases} a_i + (a_{is} - a_i) \frac{x - a_i}{c_i - a_i} & \text{for } x \in [a_i, c_i], \\ a_{i+1} + (a_{is} - a_{i+1}) \frac{x - a_{i+1}}{c_i - a_{i+1}} & \text{for } x \in [c_i, a_{i+1}], \end{cases}$$

for  $a_{is} = (1-s)a_i + sa_{i+1}$ . Then extend  $\tau'$  periodically. Owing to Lemma 3.2,  $w'(x) = p + \tilde{v}'(\tau(x))$  is a viscosity solution to

$$H(w') + \hat{V}(x) = \overline{H}_s(p) \quad \text{in } \mathbb{R}.$$

Note that  $w'$  is periodic and

$$\begin{aligned} w(1) - w(0) &= p + \int_0^1 \tilde{v}'(\tau(x)) dx \\ &= p + v(s) \left( \sum_{i=1}^{m-1} \frac{(c_i - a_i)}{s} - \sum_{i=1}^{m-1} \frac{(a_{i+1} - c_i)}{1-s} \right) \\ &= p. \end{aligned}$$

Hence  $\overline{H}(p) = \overline{H}_s(p)$ . Here  $\overline{H}$  represents the effective Hamiltonian associated with  $H(p) + \hat{V}$ .

**Step 3:** Finally, we prove that for  $H = F(|p|)$ ,  $\overline{H}_1 \equiv \overline{H}_2$  implies that (1.11) holds.

In fact, assume that  $\overline{H}_1 \equiv \overline{H}_2 = \overline{H}_s$  for some  $s \in (0, 1)$ . Clearly, there exists a unique  $s' \in (0, 1)$  such that (1.11) holds, i.e.,

$$\sum_{i=1}^{m-1} \frac{(c_i - a_i)}{s'} = \sum_{i=1}^{m-1} \frac{(a_{i+1} - c_i)}{1-s'}.$$

Due to the Step 2,  $\hat{V}$  and  $V_{s'}$  are macroscopically indistinguishable. In particular,

$$\overline{H}_1 \equiv \overline{H}_{s'}.$$

This implies  $\overline{H}_s \equiv \overline{H}_{s'}$ . So  $p_{+,s} = p_{+,s'}$ . By (3.30),  $s = s'$ . □

The following lemma was proved in Armstrong, Tran, Yu [2]. We state it here as it is needed in the proof of the above theorem. As its proof is simple, we also present it here for the sake of completeness.

**Lemma 3.1** (Generalized mean value theorem). *Suppose that  $u \in C([0, 1], \mathbb{R})$  and, for some  $a, b \in \mathbb{R}$ ,*

$$u'(0^+) = \lim_{x \rightarrow 0^+} \frac{u(x) - u(0)}{x} = a \quad \text{and} \quad u'(1^-) = \lim_{x \rightarrow 1^-} \frac{u(1) - u(x)}{1 - x} = b.$$

*Then the followings hold:*

(i) *If  $a < b$ , then for any  $c \in (a, b)$ , there exists  $x_c \in (0, 1)$  such that  $c \in D^-u(x_c)$ , i.e.,*

$$u(x) \geq u(x_c) + c(x - x_c) - o(|x - x_c|) \quad \text{for } x \in (0, 1).$$

(ii) *If  $a > b$ , then for any  $c \in (b, a)$ , there exists  $x_c \in (0, 1)$  such that  $c \in D^+u(x_c)$ , i.e.,*

$$u(x) \leq u(x_c) + c(x - x_c) + o(|x - x_c|) \quad \text{for } x \in (0, 1).$$

*Proof.* We only prove (i). For  $c \in (a, b)$ , set  $w(x) := u(x) - cx$  for  $x \in [0, 1]$ . There exists  $x_c \in [0, 1]$  such that

$$w(x_c) = \min_{x \in [0, 1]} w(x).$$

Note that  $x_c \neq 0$  and  $x_c \neq 1$  as  $c \in (a, b)$ . Thus  $x_c \in (0, 1)$ , which of course yields that  $c \in D^-u(x_c)$ . □

**Lemma 3.2.** *Suppose that  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous and  $\tau' > 0$  a.e. Assume that  $u$  is a viscosity solution of*

$$H(u') + V_1(x) = 0 \quad \text{in } \mathbb{R}.$$

*Let  $V_2(x) = V_1(\tau(x))$  and  $w'(x) = u'(\tau(x))$ . Then  $w$  is a viscosity solution to*

$$H(w') + V_2(x) = 0 \quad \text{in } \mathbb{R}.$$

*Proof.* The proof follows a straightforward change of variables. We leave it as an exercise for the readers. □

The following result is a more general version of Theorem 1.4. Before stating the theorem, we first give a definition of the potential energy  $\hat{V}_1$ .

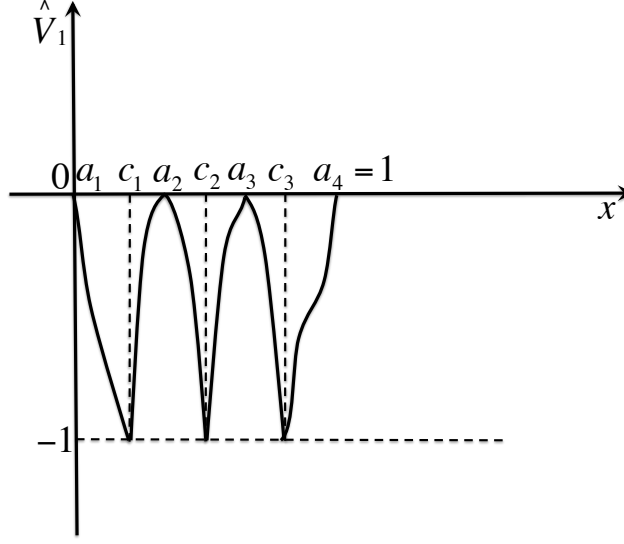
Let  $\hat{V}_1 : [0, 1] \rightarrow \mathbb{R}$  be a function oscillating between 0 and  $-1$  (see Figure 3) such that

- there exist  $0 = a_1 < c_1 < a_2 < \dots < a_{m-1} < c_{m-1} < a_m = 1$  for some  $m \geq 2$  and

$$\hat{V}_1(c_i) = -1 \quad \text{and} \quad \hat{V}_1(a_i) = 0.$$

- $\hat{V}_1$  is strictly decreasing on  $[a_i, c_i]$  and is strictly increasing on  $[c_i, a_{i+1}]$  for  $i = 1, 2, \dots, m-1$ .

Extend  $\hat{V}_1$  to  $\mathbb{R}$  in a periodic way.

FIGURE 3. Graph of  $\hat{V}_1$ 

**Theorem 3.1.** *Fix  $s \in (0, 1)$ . The two potentials  $V_s$  and  $\hat{V}_1$  are macroscopically indistinguishable if and only if, for all  $f \in C(\mathbb{R})$ ,*

$$(3.31) \quad \sum_{i=1}^{m-1} \int_{a_i}^{c_i} f(\hat{V}_1(x)) dx = \int_0^s f(V_s(x)) dx = s \int_{-1}^0 f(y) dy,$$

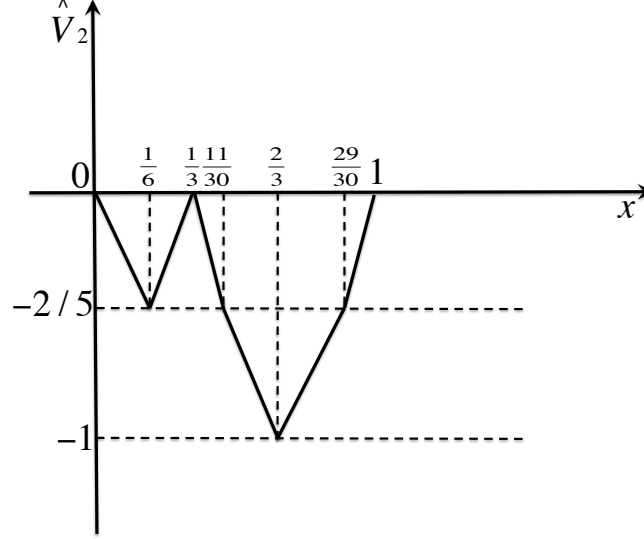
and

$$(3.32) \quad \sum_{i=1}^{m-1} \int_{c_i}^{a_{i+1}} f(\hat{V}_1(x)) dx = \int_s^1 f(V_s(x)) dx = (1-s) \int_{-1}^0 f(y) dy.$$

The proof of this theorem is basically the same as that of Theorem 1.4. We hence omit it. This result says that  $V_s$  and  $\hat{V}_1$  are macroscopically indistinguishable if and only if the distribution of the increasing parts and decreasing parts are the same respectively. Note that both  $V_s$  and  $\hat{V}_1$  are oscillating between 0 and  $-1$  here.

Let  $\hat{V}_2 : [0, 1] \rightarrow \mathbb{R}$  be a function (see Figure 4) such that

- $\hat{V}_2(0) = 0$ ,  $\hat{V}_2(\frac{1}{6}) = -\frac{2}{5}$ ,  $\hat{V}_2(\frac{1}{3}) = 0$ ,  $\hat{V}_2(\frac{11}{30}) = -\frac{2}{5}$ ,  $\hat{V}_2(\frac{2}{3}) = -1$ ,  $\hat{V}_2(\frac{29}{30}) = -\frac{2}{5}$ , and  $\hat{V}_2(1) = 0$ .
- $\hat{V}_2$  is piecewise linear in the intervals  $[0, \frac{1}{6}]$ ,  $[\frac{1}{6}, \frac{1}{3}]$ ,  $[\frac{1}{3}, \frac{11}{30}]$ ,  $[\frac{11}{30}, \frac{2}{3}]$ ,  $[\frac{2}{3}, \frac{29}{30}]$ , and  $[\frac{29}{30}, 1]$ .


 FIGURE 4. Graph of  $\hat{V}_2$ 

Extend  $\hat{V}_2$  to  $\mathbb{R}$  in a periodic way. Note that we choose  $\hat{V}_2(\frac{1}{6}) = -\frac{2}{5}$  for is just for simplicity so that we can use the nonconvex  $F$  introduced earlier. It is clear that  $V_{\frac{1}{2}}$  and  $\hat{V}_2$  have the same distribution of the increasing parts as well as the decreasing parts.

**Theorem 3.2.** *The two potentials  $V_{\frac{1}{2}}$  and  $\hat{V}_2$  are not macroscopically indistinguishable.*

*Proof.* Assume by contradiction that  $V_s$  and  $\hat{V}_2$  are macroscopically indistinguishable.

Set  $H(p) = F(|p|)$  for  $p \in \mathbb{R}$ . Recall that

$$p_{+, \frac{1}{2}} = \max\{p \geq 0 : \overline{H}_{\frac{1}{2}}(p) = 0\},$$

and in light of Step 1 in the proof of Theorem 1.4, we have that

$$\begin{aligned} p_{+, \frac{1}{2}} &= \int_{[0,1]} f_{\frac{1}{2}}(x) dx \\ &= \int_{\frac{1}{3}}^1 \psi_1(y) dy + \int_0^{\frac{1}{3}} \psi_3(y) dy + \frac{1}{2} \int_{\frac{1}{3}}^{\frac{1}{2}} (\psi_3 - \psi_1)(y) dy \\ &= \int_0^{\frac{1}{3}} \psi_3(y) dy + \frac{1}{2} \int_{\frac{1}{3}}^{\frac{1}{2}} \psi_3(y) dy + \frac{1}{2} \int_{\frac{1}{3}}^{\frac{1}{2}} \psi_1(y) dy + \int_{\frac{1}{2}}^1 \psi_1(y) dy. \end{aligned}$$

Let  $\overline{H}_2$  be the effective Hamiltonian associated with  $H(p) + \hat{V}_2(x)$ , and

$$p_{+, 2} = \max\{p \geq 0 : \overline{H}_2(p) = 0\}.$$

The same method as Step 1 in the proof of Theorem 1.4 can be used to give that

$$\begin{aligned}
p_{+,2} &= \int_0^{\frac{13}{36}} \psi_3(-\hat{V}_2(x)) dx + \int_{\frac{13}{36}}^{\frac{11}{12}} \psi_1(-\hat{V}_2(x)) dx + \int_{\frac{11}{12}}^1 \psi_3(-\hat{V}_2(x)) dx \\
&= \int_0^{\frac{1}{3}} \psi_3(y) dy + \frac{11}{12} \int_{\frac{1}{3}}^{\frac{2}{5}} \psi_3(y) dy + \frac{1}{12} \int_{\frac{1}{3}}^{\frac{2}{5}} \psi_1(y) dy \\
&\quad + \frac{1}{2} \int_{\frac{2}{5}}^{\frac{1}{2}} \psi_3(y) dy + \frac{1}{2} \int_{\frac{2}{5}}^{\frac{1}{2}} \psi_1(y) dy + \int_{\frac{1}{2}}^1 \psi_1(y) dy.
\end{aligned}$$

Therefore,

$$p_{+, \frac{1}{2}} - p_{+,2} = \frac{5}{12} \left\{ \int_{\frac{1}{3}}^{\frac{2}{5}} \psi_3(y) dy - \int_{\frac{1}{3}}^{\frac{2}{5}} \psi_1(y) dy \right\} < 0,$$

which is absurd. □

#### 4. VISCOUS CASE

For convenience, we set the diffusive constant  $d = 1$  and write  $\overline{H}_d(p) = \overline{H}(p)$  in this section.

**4.1. Connection with the Hill operator.** In this subsection, we assume  $n = 1$ . Assume  $V \in C(\mathbb{T})$ . For each  $p \in \mathbb{R}$ , the cell problem of interest is

$$(4.33) \quad -v'' + |p + v'|^2 + V(x) = \overline{H}(p) \quad \text{in } \mathbb{T}.$$

It is easy to see that

$$(4.34) \quad |p|^2 + \int_{\mathbb{T}} V dx \leq \overline{H}(p) \leq |p|^2 + \max_{\mathbb{T}} V.$$

Let us reformulate the question in the viscous case here for clarity.

**Question 4.1.** For  $i = 1, 2$ , let  $\overline{H}_i(p)$  be the viscous effective Hamiltonian associated with  $V_i$ . Assume that

$$\overline{H}_1(p) = \overline{H}_2(p) \quad \text{for all } p \in \mathbb{R}.$$

What can we say about  $V_1$  and  $V_2$ ?

For  $\lambda \in \mathbb{C}$ , let  $w_1$  solve

$$\begin{cases} -w_1'' = (\lambda + V)w_1 & \text{in } (0, 1), \\ w_1(0) = 1, \quad w_1'(0) = 0, \end{cases}$$

and  $w_2$  solve

$$\begin{cases} -w_2'' = (\lambda + V)w_2 & \text{in } (0, 1), \\ w_2(0) = 0, \quad w_2'(0) = 1. \end{cases}$$

Denote

$$M = \begin{pmatrix} w_1(1) & w_2(1) \\ w_1'(1) & w_2'(1) \end{pmatrix}.$$



It is clear that  $\det(M) = 1$ , and therefore,  $A$  has two eigenvalues  $\theta$  and  $\frac{1}{\theta}$ . Denote

$$\Delta(\lambda) = \theta + \frac{1}{\theta} = w_1(1) + w_2'(1),$$

which is the so called *discriminant* associated with the Hill operator  $Q = -\frac{d^2}{dx^2} - V(x)$ . See page 295 in [17]. One can easily show that  $\Delta(\lambda)$  is an entire function. Obviously,  $\Delta(\lambda) \in \mathbb{R}$  if  $\lambda \in \mathbb{R}$ .

**Lemma 4.1.** *For  $-\lambda \geq \min_{\mathbb{R}} \overline{H}$ ,*

$$\{p \in \mathbb{R} : \overline{H}(p) = -\lambda\} = \log \left( \frac{\Delta(\lambda) \pm \sqrt{\Delta^2(\lambda) - 4}}{2} \right).$$

*Proof.* Let  $\overline{H}(p) = -\lambda$  and  $v$  be a solution to cell problem (4.33). Denote

$$w = e^{-(px+v)}.$$

Then  $w$  satisfies that

$$\begin{cases} -w'' - Vw = \lambda w & \text{in } \mathbb{R} \\ w(1) = e^{-p}w(0), \quad w'(1) = e^{-p}w'(0). \end{cases}$$

Assume that  $w = a_1w_1 + a_2w_2$  for  $a_1, a_2 \in \mathbb{R}$ . Then an easy calculation shows that

$$MA = e^{-p}A,$$

for  $A = (a_1, a_2)^T$ . So  $e^{-p}$  is an eigenvalue of  $M$ .  $\square$

Since  $\Delta(\lambda)$  is an entire function, our inverse problem (Question 4.1) is equivalent to the following question:

**Question 4.2.** *For  $i = 1, 2$ , let  $\Delta_i(\lambda)$  be the discriminant associated with  $V_i$ . Assume that*

$$\Delta_1(\lambda) = \Delta_2(\lambda) \quad \text{for all } \lambda \in \mathbb{R}.$$

*What can we say about  $V_1$  and  $V_2$ ?*

It is known that the discriminant is determined by the spectrum of the Hill operator:  $\Delta_1(\lambda) \equiv \Delta_2(\lambda)$  if and only if the following two Hill operators

$$L_1 = -\frac{d^2}{dx^2} - V_1 \quad \text{and} \quad L_2 = -\frac{d^2}{dx^2} - V_2$$

have the same eigenvalues. See Theorem XIII.92 in [17].

**Remark 4.1.** *Unlike the inviscid one dimensional case, that  $V_1$  and  $V_2$  have the same distribution is neither a necessary nor a sufficient condition for  $\Delta_1 \equiv \Delta_2$  (equivalently,  $\overline{H}_1 = \overline{H}_2$ .)*

*To see the non-sufficiency is quite simple. As suggested by Elena Kosygina, fix  $V \in C(\mathbb{T})$  and look at  $V_m = V(mx)$ . Clearly,  $V_m$  and  $V$  have the same distribution. However, as  $m \rightarrow +\infty$ , the viscous effective Hamiltonian associated with  $V_m$  converges to the inviscid effective Hamiltonian associated with  $V$ . The inviscid effective Hamiltonian is always larger than the viscous one for non-constant  $V$ .*

*The non-necessity is more tricky. It is known that the KdV equation preserves the discriminant. More precisely, if  $q(x, t)$  is a smooth space periodic solution to the KdV equation*

$$q_t + qq_x + q_{xxx} = 0,$$

*then the spectrum (or the discriminant) associated with the Hill operator*

$$L = -\frac{d^2}{dx^2} - \frac{1}{6}q(\cdot, t)$$

*is independent of  $t$ . See [13, 14] for instance. However, the distribution of  $q(\cdot, t)$  is not invariant under the KdV equation. In fact, without involving derivatives of  $q$ , only the quantities  $\int_{\mathbb{T}} q(x, t) dx$  and  $\int_{\mathbb{T}} q^2(x, t) dx$  are conserved. Hence (3) in Theorem 1.6 is optimal.*

#### 4.2. Proof of Theorem 1.5.

*Proof.* We first prove (1). Owing to part (1) of Theorem 1.6, we have that

$$\int_{\mathbb{T}^n} V dx = 0.$$

Since  $H$  is superlinear, we may choose  $p_0 \in \mathbb{R}^n$  such that

$$H(p_0) - \sqrt{1 + |p_0|^2} = \min_{\mathbb{R}^n} \left( H(p) - \sqrt{1 + |p|^2} \right) = h_0.$$

Denote  $\tilde{H}(p) = \sqrt{1 + |p|^2} + h_0$ . Then  $H(p) \geq \tilde{H}(p)$  and  $H(p_0) = \tilde{H}(p_0)$ . Clearly,

$$H = \overline{H} \geq \tilde{H} \geq \tilde{H} + \int_{\mathbb{T}^n} V dx = \tilde{H}.$$

Here  $\overline{\tilde{H}}$  represents the viscous effective Hamiltonian associated with  $\tilde{H} + V$  with  $d = 1$ . Accordingly,

$$\overline{\tilde{H}}(p_0) = \tilde{H}(p_0) + \int_{\mathbb{T}^n} V dx = \tilde{H}(p_0).$$

Let  $v_0 \in C^\infty(\mathbb{T}^n)$  be a solution to

$$-\Delta v_0 + \tilde{H}(p_0 + Dv_0) + V = \overline{\tilde{H}}(p_0) = \tilde{H}(p_0) \quad \text{in } \mathbb{T}^n.$$

Taking integration over  $\mathbb{T}^n$  and using the strict convexity of  $\tilde{H}$ , we obtain that  $Dv_0 \equiv 0$ . Hence  $V \equiv 0$ .

Next we prove (2). Let  $v = v(x, p) \in C^\infty(\mathbb{T}^n)$  be the unique solution to

$$\begin{cases} -\Delta v + |p + Dv|^2 + V(x) = \overline{H}(p) & \text{in } \mathbb{T}^n, \\ \int_{\mathbb{T}^n} v dx = 0. \end{cases}$$

Then  $w = e^{-v}$  satisfies that

$$\Delta w - 2p \cdot Dw + V(x)w = (\overline{H}(p) - |p|^2)w \quad \text{in } \mathbb{T}^n.$$

For  $w_0(x) = w(x, 0)$ , it is clear that

$$(4.35) \quad \Delta w_0 + V(x)w_0 = 0 \quad \text{in } \mathbb{T}^n.$$

Taking partial derivatives of  $w(x, p)$  with respect to  $p_1$  and evaluating at  $p = 0$ , we obtain that, for  $w_1(x) = w_{p_1}(x, 0)$  and  $w_2(x) = w_{p_1 p_1}(x, 0)$ ,

$$(4.36) \quad \Delta w_1 + V(x)w_1 = 2w_{x_1} \quad \text{in } \mathbb{T}^n$$

and

$$(4.37) \quad \Delta w_2 + V(x)w_2 = 4w_{x_1 p_1} \quad \text{in } \mathbb{T}^n.$$

Accordingly, for  $Q = [0, 1]^n$ ,

$$\int_Q w_0 w_{x_1 p_1}(x, 0) dx = 0.$$

This implies that  $\int_Q w_1 w_{x_1}(x, 0) dx = 0$ . In light of (4.36), one deduces that

$$\int_Q (|Dw_1|^2 - V(x)w_1^2) dx = 0.$$

Since  $w_0 > 0$  is the principle eigenfunction of the symmetric operator  $L = -\Delta - V$ , we have that

$$0 = \min_{\phi \in H^1(\mathbb{T}^n)} \int_Q (|D\phi|^2 - V(x)\phi^2) dx.$$

Hence  $w_1 = \lambda w_0$  for some  $\lambda \in \mathbb{R}$ , which gives that  $w_{x_1} \equiv 0$ . Similarly, we can show that  $w_{x_i}(x, 0) \equiv 0$  for  $i \geq 2$ .

Therefore,  $w_0$  is a positive constant and  $V \equiv 0$ .

□

#### 4.3. Proof of Theorem 1.6.

*Proof.* The proof of (1.14) is essentially the same as (1.5). Since  $Q$  satisfies a Diophantine condition, there exists a unique smooth periodic solution  $v$  (up to an additive constant) to

$$Q \cdot Dv = a_1 - V_1 \quad \text{in } \mathbb{T}^n,$$

for  $a_1 = \int_{\mathbb{T}^n} V_1 dx$ . Then it is easy to see that for  $v_\lambda = \frac{v}{\lambda}$ ,

$$-\Delta v_\lambda + H(\lambda P_\lambda + Dv_\lambda) + V(x) = H(\lambda P_\lambda) + a_1 + O\left(\frac{1}{\lambda}\right) \quad \text{in } \mathbb{T}^n.$$

Let  $w_\lambda \in C^\infty(\mathbb{T}^n)$  be a viscosity solution to

$$-\Delta w_\lambda + H(\lambda P_\lambda + Dw_\lambda) + V(x) = \overline{H}_1(\lambda P_\lambda) \quad \text{in } \mathbb{T}^n.$$

By looking at the places where  $w_\lambda - v_\lambda$  attains its maximum and minimum, we get that

$$\overline{H}_1(\lambda P_\lambda) = H(\lambda P_\lambda) + a_1 + O\left(\frac{1}{\lambda}\right).$$

Next we prove (2). Let  $Q$  be a unit vector satisfying a Diophantine condition. For  $i = 1, 2$ , we can explicitly solve the following equations in  $\mathbb{T}^n$  by computing Fourier coefficients

$$(4.38) \quad \begin{cases} 2Q \cdot Dv_{i1} = a_{i1} - V_i \\ 2Q \cdot Dv_{i2} = \Delta v_{i1} \\ 2Q \cdot Dv_{i3} = a_{i2} - |Dv_{i1}|^2 + \Delta v_{i2}. \end{cases}$$

Here  $a_{i1} = \int_{\mathbb{T}^n} V_i dx$  and  $a_{i2} = \int_{\mathbb{T}^n} |Dv_1|^2 dx$ . Then for  $\varepsilon > 0$ ,  $v_{i\varepsilon} = \varepsilon^2 v_{i1} + \varepsilon^3 v_{i2} + \varepsilon^4 v_{i3}$  satisfy

$$-\varepsilon \Delta v_{i\varepsilon} + |Q + Dv_{i\varepsilon}|^2 + \varepsilon^2 V_i = |Q|^2 + \varepsilon a_{i1} + \varepsilon^4 a_{i2} + O(\varepsilon^5).$$

Suppose that  $w_{i\varepsilon} \in C^\infty(\mathbb{T}^n)$  is a solution to

$$-\varepsilon \Delta w_{i\varepsilon} + |Q + Dw_{i\varepsilon}|^2 + \varepsilon^2 V_i = \overline{H}_{i\varepsilon}(Q) \quad \text{in } \mathbb{R}.$$

Here  $\overline{H}_{i\varepsilon}(Q) = \varepsilon^2 \overline{H}_i(\frac{Q}{\varepsilon})$  is the viscous effective Hamiltonian associated with  $|p|^2 + \varepsilon V_i$  with  $d = \varepsilon$ . By looking at places where  $v_{i\varepsilon} - w_{i\varepsilon}$  attains its maximum and minimum, we derive that

$$\overline{H}_{i\varepsilon}(Q) = |Q|^2 + \varepsilon^2 a_{i1} + \varepsilon^4 a_{i2} + O(\varepsilon^5).$$

To finish the proof, it suffices to show that for any  $Q$  satisfying a Diophantine condition

$$\int_{\mathbb{T}^n} |Dv_{11}|^2 dx = \int_{\mathbb{T}^n} |Dv_{21}|^2 dx,$$

then

$$\int_{\mathbb{T}^n} |V_1|^2 dx = \int_{\mathbb{T}^n} |V_2|^2 dx.$$

From here, the proof goes exactly the same as that of (3) in Theorem 1.2.

Finally, (3) follows immediately from the fact that

$$v'_{i1} = \frac{1}{2}(a_{i1} - V_i) \quad \text{in } \mathbb{T},$$

for  $n = 1$ ,  $Q = 1$  and  $i = 1, 2$ . □

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